Polynomial Functions and their Graphs

\[ q(x) = (-0.029x^2 - 0.09486x^1 + 0.2358 + 0.84) \]

**Definition of a Polynomial Function**

Let \( n \) be a nonnegative integer and let \( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \) be real numbers, with \( a_n \neq 0 \). The function defined by

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0
\]

is called a polynomial function of degree \( n \). The number \( a_n \), the coefficient of the variable to the highest power, is called the leading coefficient.

\[
f(x) = -3x^5 + \sqrt{2}x^2 + 5
\]

Degree 5 Yes

\[
f(x) = -3x^4(x - 2)(x + 3)
\]

Yes degree 6

\[
f(x) = -3\sqrt{x} + \sqrt{2}x^2 + 5
\]

\[
x^\frac{3}{2}
\]

No

\[
f(x) = \frac{-3}{x^2} + \sqrt{2}x^2 + 5
\]

\[
x^2
\]

No

\[
f(x) = -3x^4(x - 2)(x + 3)
\]
Polynomial functions of degree 2 or higher have graphs that are smooth and continuous. By smooth, we mean that the graphs contain only rounded curves with no sharp corners. By continuous, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system.

The behavior of the graph of a function to the far left or the far right is called its end behavior. Although the graph of a polynomial function may have intervals where it increases or decreases, the graph will eventually rise or fall without bound as it moves far to the left or far to the right.

Let's look at the end behavior of some polynomials. Do you see a pattern? What if the graphs had a negative a value?
\[ g(x) = 3x^3 \quad f(x) = 3x^3 - 4x^2 - 4x + 2 \]

\[ g(x) = 2x^4 \quad f(x) = 2x^4 + 2x^3 - 22x^2 - 18x + 15 \]
The Leading Coefficient Test

As \( x \) increases or decreases without bound, the graph of the polynomial function

\[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (a_n \neq 0) \]

eventually rises or falls. In particular,

1. For \( n \) odd:
   - If the leading coefficient is positive, the graph falls to the left and rises to the right. (\( \downarrow, \uparrow \))
   - If the leading coefficient is negative, the graph rises to the left and falls to the right. (\( \uparrow, \downarrow \))

2. For \( n \) even:
   - If the leading coefficient is positive, the graph rises to the left and rises to the right. (\( \uparrow, \uparrow \))
   - If the leading coefficient is negative, the graph falls to the left and falls to the right. (\( \downarrow, \downarrow \))

Use the Leading Coefficient Test to determine the end behavior of the graphs below:

\[ f(x) = x^4 - 4x^2 \quad \text{rise left, rise right} \]
\[ n = \text{even} \quad a_n > 0 \]

\[ f(x) = x^3 + 3x^2 - x - 3 \quad \text{fall left, rise right} \]
\[ n = \text{odd} \quad a_n > 0 \]

\[ f(x) = -4x^3(x - 1)^2(x + 5) \quad \text{fall left, fall right} \]
\[ n = \text{even} \quad a_n < 0 \]
Is this the complete graph of the function 
\[ f(x) = x^3 + 13x^2 + 10x - 4 \] ?

It is shown in the standard default window.

To find the zeros of the polynomial function - factor

\[ f(x) = x^3 - x^2 - 6x \]
\[ f(x) = x(x^2 - x - 6) \]
\[ f(x) = x(x - 3)(x + 2) \]

\( x = 0, x = 3, x = -2 \)

\( y\)-int = 0, end behavior like \( x^3 \)

Now sketch a graph of the function.
\( f(x) = 5x^3 - 5x^2 - 10x \)

Find the zeros (roots, solutions)

\[
0 = 5x(x^2 - x - 2)
0 = 5x(x - 2)(x + 1)
\]

\[
x = 0 \quad x = 2 \quad x = -1
\]

\( y \)-int \( 0 \)

End behavior \( y = 5x^3 \) (fall left rise right)

\[
\begin{array}{c|c}
 x & y \\
 1 & -10 \\
\end{array}
\]

\( f(x) = x^4 + 4x^3 + 4x^2 \)

\[
x^2(x^2 + 4x + 4)
\]

\[
x^2(x + 2)(x + 2)
\]

\[
x(x + 2)(x + 2)
\]

\( x = 0 \) multiplicity of 2

\( x = -2 \) multiplicity of 2

If \( f \) is a polynomial function and \( (x - c)^m \) is a factor of \( f \) but \( (x - c)^{m+1} \) is not, then \( c \) is a zero of multiplicity \( m \) of \( f \).
If a polynomial function $f$ has a real zero $c$ of odd multiplicity, then the graph of $f$ crosses the $x$-axis at $(c, 0)$ and the value of $f$ changes sign at $x = c$.

If a polynomial function $f$ has a real zero $c$ of even multiplicity, then the graph of $f$ does not cross the $x$-axis at $(c, 0)$ and the value of $f$ does not change sign at $x = c$.
f(x) = 7(x - 3)(x + 5)
\[ \frac{7(-4)(4)}{7(0-3)(0+5)} = \frac{-105}{-105} \]
zeros:
3 mult 1 cross
-5 mult 1 cross
y-int
\[ 7(0-3)(0+5) = -105 \]
end behavior
\[ x^2 \text{ rise left rise right} \]
sketch the graph

f(x) = (x - 1)^3(x + 4)^2
zeros:
1 mult of 2 cross
-4 mult of 2 touch
y int
\[ (1-1)^3(1+4)^2 = -16 \]
end behavior
\[ x^2 \text{ fall left rise right} \]
sketch graph
Recall that all polynomial functions are continuous (no jumps, no holes) smooth curves.

Intermediate Value theorem:
If \( f(a) \) and \( f(b) \) have opposite signs (one is negative and one is positive), then \( f(c)=0 \) for some number \( c \) in \( [a,b] \).

Can use this with graphing calculator and the trace button. Watch for sign change to find zeros.

The graph of \( x^5 - 6x^3 + 8x + 1 \) has four smooth turning points.

If \( f \) is a polynomial function of degree \( n \), then the graph of \( f \) has at most \( n-1 \) turning points (relative maxima or minima).
Write an equation, expressed as the product of factors, of a polynomial function for the graph.

\[(x+3)^{\text{odd}}(x+1)^{\text{odd}}(x-2)^{\text{even}}\]

Assignment
Lesson 2.3

3, 7, 11-23 odd,
27-31 odd, 37,
41, 45, 53, 55,
63, 69