Lesson 2.6 Day 2  Graphing Rational Functions

Rational functions that are not transformations of \( f(x) = \frac{1}{x} \) or \( f(x) = \frac{1}{x^2} \) can be graphed using the following procedure:

**Strategy for Graphing a Rational Function**

The following strategy can be used to graph

\[
f(x) = \frac{p(x)}{q(x)},
\]

where \( p \) and \( q \) are polynomial functions with no common factors.

1. Determine whether the graph of \( f \) has symmetry.
   \[
   f(-x) = f(x): \quad \text{y-axis symmetry}
   \]
   \[
   f(-x) = -f(x): \quad \text{origin symmetry}
   \]

2. Find the \( y \)-intercept (if there is one) by evaluating \( f(0) \).
3. Find the \( x \)-intercepts (if there are any) by solving the equation \( p(x) = 0 \).
4. Find any vertical asymptote(s) by solving the equation \( q(x) = 0 \).
5. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each \( x \)-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

Graph the rational function \( f(x) = \frac{2x-1}{x-1} \)

**Step 1: Determine symmetry**

\[
f(-x) = \frac{2(-x)-1}{(-x)-1} = \frac{-2x-1}{-x-1}
\]

Neither

\[
-2x-1
\]
Graph the rational function \( f(x) = \frac{2x-1}{x-1} \)

Step 2: Find the y intercept
(put 0 in for x)
\[
y = \frac{2(0)-1}{0-1} = \frac{-1}{-1} = 1 \quad (0, 1)
\]

Step 3: Find the x-intercepts (put 0 in for f(x))
Set numerator equal to 0
\[
0 = \frac{2x-1}{x-1} \quad 2x-1 = 0 \quad x = \frac{1}{2} \quad \left(\frac{1}{2}, 0\right)
\]

Graph the rational function \( f(x) = \frac{2x-1}{x-1} \)

Step 4: Find the vertical asymptotes
Set denominator = 0
\[
x-1 = 0 \quad x = 1 \quad \text{V. A. at} \quad x = 1
\]

Step 5: Find the horizontal asymptotes
Look at the degree of numerator and denominator
\( n=m \)
\[
\text{H. A. at} \quad y = \frac{2}{1} \quad y = 2
\]
Graph the rational function \( f(x) = \frac{2x - 1}{x - 1} \)

Step 6: Plot points between and beyond each x-intercept and vertical asymptote

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>( \frac{3}{4} )</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{2x - 1}{x - 1} )</td>
<td>( \frac{5}{3} )</td>
<td>( \frac{3}{2} )</td>
<td>-2</td>
<td>3</td>
<td>( \frac{7}{3} )</td>
</tr>
</tbody>
</table>

Graph the rational function \( f(x) = \frac{2x - 1}{x - 1} \)

Step 7: Graph the function

- Horizontal asymptote: \( y = 2 \)
- Vertical asymptote: \( x = 1 \)

**Figure 2.34** Preparing to graph the rational function \( f(x) = \frac{2x - 1}{x - 1} \)

**Figure 2.35** The graph of \( f(x) = \frac{2x - 1}{x - 1} \)

\[ \lim_{x \to -\infty} = 2 \]
\[ \lim_{x \to +\infty} = 2 \]

\[ \lim_{x \to 1^-} = -\infty \]
\[ \lim_{x \to 1^+} = +\infty \]
Look at examples 6 and 7 from your book. All seven steps are shown for the process of graphing each function.

End behavior asymptotes do not have to be horizontal lines. In fact, they don't even have to be lines.

**Slant asymptotes.**

\[
f(x) = \frac{x^2 + 1}{x - 1}
\]

If the degree of the numerator is one more than the degree of the denominator the function will have a slant asymptote.
To find the equation of the slant asymptote, divide the rational function.

\[ f(x) = \frac{x^2 + 1}{x - 1} \]

\[
\begin{array}{c|cc}
1 & 1 & 0 & 1 \\
\hline \\
& 1 & 1 & 1 \\
\hline \\
& 1 & 1 & 2 \\
\end{array}
\]

\textcolor{red}{x + 1} \ r 2

equation of slant asymptote is \( y = x + 1 \)

**Characteristics of rational functions.**

\[
\frac{f(x)}{g(x)} \quad \text{n - degree} \quad \text{m - degree}
\]

**End behavior asymptote:**

If \( n < m \), the end behavior asymptote is the horizontal asymptote \( y = 0 \).

If \( n = m \), the end behavior asymptote is the horizontal asymptote \( y = \frac{\text{leading coefficient of } f(x)}{\text{leading coefficient of } g(x)} \).

If \( n > m \), the end behavior asymptote is the quotient polynomial function \( y = q(x) \).

There is no horizontal asymptote.

**Vertical asymptotes:** These occur at the zeros of the denominator, provided that the zeros are not also zeros of the numerator of equal or greater multiplicity.

**x-intercepts:** These occur at the zeros of the numerator, which are not also zeros of the denominator.

**y-intercept:** This is the value of \( f(0) \), if defined.

\[ \text{put 0 in for } x \]
\[ f(x) = \frac{x^3}{x^2 - 9} \]

\[ x^2 - 9 = 0 \]
\[ (x - 3)(x + 3) = 0 \]
\[ x = 3 \quad x = -3 \]

Vert. Asym. \( x = 3, x = -3 \)
End behavior asym. \( y = x \) (slant asym)
X-intercepts = 0 Set numerator = to 0
Y-intercepts = 0 Put 0 in for \( x \)

\[ \lim_{x \to -\infty} = -\infty \]
\[ \lim_{x \to \infty} = \infty \]

Slant asymptote:
End behavior asymptote

\[ f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1} \]

\[ \begin{array}{c|cccc}
  & 1 & -3 & 3 & 1 \\
\hline
 1 & 1 & -3 & 3 & 1 \\
\end{array} \]

\[ \frac{1}{-2} = -1 \]

Vert. Asym. \( x = 1 \)
End behavior asym. \( y = x^2 - 2x + 1 \)
X-int.: see graph
Y-int.: put 0 in for \( x \)
\[ \frac{1}{1} = -1 \]
Suppose a company that manufactures a robotic exoskeleton has a fixed monthly cost of $1,000,000 and that it cost $5000 to produce each robotic system.

a. Write the cost function, $C$, of producing $x$ robotic systems.
   \[ C = 1,000,000 + 5000x \]

b. Write the average cost function $\overline{C}$, of producing $x$ robotic systems.
   \[ \overline{C} = \frac{1,000,000 + 5000x}{x} \]

c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, $\overline{C}(100,000)$.

d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.

\[ y = \frac{5000}{x} \quad y = 5000 \]
A commuter drove to work a distance of 40 miles and then returned again on the same route. The average velocity on the return trip was 30 miles per hour faster than the average velocity on the outgoing trip. Express the total time required to complete the round trip, $T$, as a function of the average velocity on the outgoing trip, $x$.

\[
d = r \cdot t
\]

\[
d = \text{velocity} \times \text{time}
\]

\[
t = \frac{\text{distance}}{\text{velocity}}
\]

\[
T(x) = \frac{40}{x} + \frac{40}{x + 30}
\]

Assignment
Lesson 2.6 Day 2

57, 61, 63, 73, 75, 89, 91, 99, 120-123