Kindness Week

High Five a Stranger

4.2 Trigonometry Functions: The Unit Circle

Objectives
1. Use a unit circle to define trigonometric functions of real numbers.
2. Recognize the domain and range of sine and cosine functions.
3. Find exact values of the trigonometric functions at \( \frac{\pi}{4} \).
4. Use even and odd trigonometric functions.
5. Recognize and use fundamental identities.
6. Use periodic properties.
7. Evaluate trigonometric functions with a calculator.
The Unit Circle

A unit circle is a circle of radius 1, with its center at the origin of a rectangular coordinate system. The equation of this unit circle is $x^2 + y^2 = 1$. Figure 4.19 shows a unit circle with a central angle measuring $t$ radians.

We can use the formula for the length of a circular arc, $s = r\theta$, to find the length of the intercepted arc.

Thus, the length of the intercepted arc is $t$. This is also the radian measure of the central angle. Thus, in a unit circle, the radian measure of the central angle is equal to the length of the intercepted arc. Both are given by the same real number $t$.

$$ (x - h)^2 + (y - k)^2 = r^2 $$

Definitions of the Trigonometric Functions in Terms of a Unit Circle

If $t$ is a real number and $P = (x, y)$ is a point on the unit circle that corresponds to $t$, then

- $\sin t = y$
- $\cos t = x$
- $\tan t = \frac{y}{x}, x \neq 0$
- $\csc t = \frac{1}{y}, y \neq 0$
- $\sec t = \frac{1}{x}, x \neq 0$
- $\cot t = \frac{x}{y}, y \neq 0$. 

\[ \frac{\sin t}{\cos t} = \tan t \]
Reciprocal & Quotient Identities

<table>
<thead>
<tr>
<th>Reciprocal Identities</th>
<th>Quotient Identities</th>
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<tbody>
<tr>
<td>( \sin t = \frac{1}{\csc t} )</td>
<td>( \tan t = \frac{\sin t}{\cos t} )</td>
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<tr>
<td>( \cos t = \frac{1}{\sec t} )</td>
<td>( \cot t = \frac{1}{\tan t} )</td>
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Given \( \sin t = \frac{2}{3} \) and \( \cos t = \frac{\sqrt{5}}{3} \)
find the four remaining trig functions

\[
\sin t = \frac{2}{3}, \quad \csc t = \frac{3}{2}, \\
\cos t = \frac{\sqrt{5}}{3}, \quad \sec t = \frac{3}{\sqrt{5}}, \\
\tan t = \frac{2}{\sqrt{5}}, \quad \cot t = \frac{\sqrt{5}}{2}
\]

\[
\tan t = \frac{\frac{2}{\sqrt{5}}}{\frac{\sqrt{5}}{3}} = \frac{2 \cdot 3}{\sqrt{5} \cdot \sqrt{5}} = \frac{6}{5} = \frac{2\sqrt{5}}{5}
\]

A point \( P(x, y) \) is shown on the unit circle corresponding to a real number \( t \).
Find the values of the trig functions at \( t \).

\[
\sin t = \frac{-12}{13}, \quad \csc t = \frac{-13}{12}, \\
\cos t = \frac{-5}{13}, \quad \sec t = \frac{-13}{5}, \\
\tan t = \frac{12}{5}, \quad \cot t = \frac{5}{12}
\]

\[
\tan t = \frac{-12}{-\frac{5}{13}} = \frac{-12 \cdot 13}{5} = \frac{12}{5}
\]
**Domain & Range of Sine and Cosine Functions**

The Domain and Range of the Sine and Cosine Functions
The domain of the sine function and the cosine function is \((-\infty, \infty)\), the set of all real numbers. The range of these functions is \([-1, 1]\), the set of all real numbers from \(-1\) to 1, inclusive.

**Even and Odd Trigonometric Functions**

\[
\cos(-t) = \cos(t) \quad \text{Even}
\]

\[
\sin(-t) = -\sin(t) \quad \text{Odd}
\]

**Even and Odd Trigonometric Functions**

The cosine and secant functions are **even**.
\[
\cos(-t) = \cos t \quad \text{sec}(-t) = \sec t
\]

The sine, cosecant, tangent, and cotangent functions are **odd**.
\[
\sin(-t) = -\sin t \quad \csc(-t) = -\csc t
\]
\[
\tan(-t) = -\tan t \quad \cot(-t) = -\cot t
\]
Use Even and Odd Trigonometric Properties

\[
\cos \frac{\pi}{3} = \frac{1}{2} \quad \text{even}
\]
\[
\cos \left( -\frac{\pi}{3} \right) = \frac{1}{2}
\]

\[
\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3} \quad \text{odd}
\]
\[
\csc \left( -\frac{\pi}{3} \right) = -\frac{2\sqrt{3}}{3}
\]

**Even and Odd Trigonometric Functions**

The cosine and secant functions are *even*.

\[
\cos(-t) = \cos t \quad \sec(-t) = \sec t
\]

The sine, cosecant, tangent, and cotangent functions are *odd*.

\[
\sin(-t) = -\sin t \quad \csc(-t) = -\csc t
\]
\[
\tan(-t) = -\tan t \quad \cot(-t) = -\cot t
\]
Pythagorean Identities

\[ x^2 + y^2 = 1 \]
\[ \cos^2 t + \sin^2 t = 1 \]

Pythagorean Identities

\[ \sin^2 t + \cos^2 t = 1 \]

1 + \tan^2 t = \sec^2 t 
1 + \cot^2 t = \csc^2 t 

\[ 0 < t < \frac{\pi}{2} \]
\[ \sin t = \frac{\sqrt{8}}{5} \]
\[ \cos t = \frac{3}{5} \]

\[ \sin^2 t + \cos^2 t = 1 \]
\[ \left( \frac{\sqrt{8}}{5} \right)^2 + \cos^2 t = 1 \]
\[ \frac{8}{25} + \cos^2 t = 1 \]
\[ \cos^2 t = \frac{25}{25} - \frac{8}{25} \]
\[ \cos^2 t = \frac{17}{25} \]
\[ \cos t = \frac{\sqrt{17}}{5} \]

Periodic Functions

Definition of a Periodic Function
A function \( f \) is periodic if there exists a positive number \( p \) such that
\[ f(t + p) = f(t) \]
for all \( t \) in the domain of \( f \). The smallest positive number \( p \) for which \( f \) is periodic is called the period of \( f \).

Periodic Properties of the Sine and Cosine Functions

\[ \sin(t + 2\pi) = \sin t \quad \text{and} \quad \cos(t + 2\pi) = \cos t \]

The sine and cosine functions are periodic functions and have period \( 2\pi \).

Periodic Properties of the Tangent and Cotangent Functions

\[ \tan(t + \pi) = \tan t \quad \text{and} \quad \cot(t + \pi) = \cot t \]

The tangent and cotangent functions are periodic functions and have period \( \pi \).
Repetitive Behavior of the Sine, Cosine, and Tangent Functions
For any integer $n$ and real number $t$,
\[
\sin(t + 2\pi n) = \sin t, \quad \cos(t + 2\pi n) = \cos t, \quad \text{and} \quad \tan(t + \pi n) = \tan t.
\]

Evaluate without a calculator

\[
\cos \left( \frac{3\pi}{4} + 50\pi \right) = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}
\]
\[
\sin \left( -\frac{5\pi}{4} - 190\pi \right) = \sin \frac{5\pi}{4} = \frac{\sqrt{2}}{2}
\]
\[
-\cot \left( \frac{3\pi}{4} - 1011\pi \right) = -\cot \frac{3\pi}{4} = (-1) = 1
\]
Evaluate with a calculator
Find the value to four decimal places

\[
\cot\left(\frac{9\pi}{4}\right) = \frac{1}{\tan\left(\frac{9\pi}{4}\right)}
\]

\[
\sec\left(-\frac{3\pi}{4}\right) = \cos\left(-\frac{3\pi}{4}\right) \approx -1.41
\]

\[
\csc 1.2 = \sin(1.2) \approx 1.07
\]
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1, 3, 25 - 37 odd,
65 - 69 odd,
74 - 77, 80, 81, 84