Lesson 6.5

Complex Numbers in Polar Form

Objectives
1. Plot complex numbers in the complex plane.
2. Find the absolute value of a complex number.
3. Write complex numbers in polar form.
4. Convert a complex number from polar to rectangular form.
5. Find products of complex numbers in polar form.
6. Find quotients of complex numbers in polar form.
7. Find powers of complex numbers in polar form.
8. Find roots of complex numbers in polar form.

One of the new frontiers of mathematics suggests that there is an underlying order in things that appear to be random. Chaotic behavior in the mathematical sense does not mean a complete lack of form. In mathematics, chaos is used to describe something that appears to be random but is not actually random. The hidden structure of chaotic events can be studied by opening up graphing to include complex numbers.

The Complex Plane

The complex plane, often referred to as the rectangular form, is a two-dimensional plane where each point represents a complex number. The horizontal axis represents the real part, while the vertical axis represents the imaginary part.

Plot each complex number in the complex plane:

\[ z = 3 + 4i \]
\[ z = -1 - 2i \]
\[ z = -3 + 2i \]
\[ z = -4i \]
\[ 0 + 4i \]
The absolute value of a complex number

Recall that the absolute value of a real number is its distance from 0 on a number line. The absolute value of the complex number \( z = a + bi \), denoted by \( |z| \), is the distance from the origin to the point \( z \) in the complex plane.

Determine the absolute value of \( z = 3 + 4i \) and \( z = -1 - 2i \)

\[
|z_1| = \sqrt{3^2 + 4^2} = 5 \\
|z_2| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}
\]

The Absolute Value of a Complex Number

The absolute value of the complex number \( a + bi \) is

\[
|z| = |a + bi| = \sqrt{a^2 + b^2}.
\]

Polar Form of a Complex Number

\[
z = a + bi \\
\Rightarrow \quad \text{rectangular} \quad z = r \cos \theta + (r \sin \theta)i \\
\Rightarrow \quad \text{polar} \quad z = r(\cos \theta + i \sin \theta)
\]

Real

\( (x, y) \rightarrow \text{rectangular} \)

\( [r, \theta] \rightarrow \text{polar} \)

Complex

\( a + bi \rightarrow \text{rectangular} \)

\( r(\cos \theta + i \sin \theta) \rightarrow \text{polar} \)

modulus

argument
Write $z = -1 - i\sqrt{3}$ in polar form

$r^2 = \sqrt{a^2 + b^2}$
$r^2 = (1)^2 + (\sqrt{3})^2$
$r^2 = 4$
$r = \pm 2$

$\tan \theta = \frac{b}{a}$
$\tan \theta = \frac{-\sqrt{3}}{1}$
$\theta = \frac{4\pi}{3}$

$z = 2 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)$

Write $4(\cos 30^\circ + i \sin 30^\circ)$ in rectangular form

$a = r \cos \theta$
$a = 4 \cos 30^\circ = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$b = r \sin \theta$
$b = 4 \sin 30^\circ = 4 \cdot \frac{1}{2} = 2$

$z = 2\sqrt{3} + 2i$
Convert to polar form

rectangular form \( 10 + 0i \)

polar form
\[
\begin{align*}
 r^2 &= \sqrt{10^2 + 0^2} = r = \pm 10 \\
 \tan \theta &= \frac{0}{10} = 0 \text{ or } \pi \\
 = 10 \left( \cos 0^\circ + i \sin 0^\circ \right)
\end{align*}
\]

rectangular form \( 0 + 3i \)

polar form
\[
\begin{align*}
 r^2 &= \sqrt{0^2 + 3^2} = \pm 3 \\
 \tan \theta &= \frac{3}{0} = \text{ undefined} \quad \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\
 = 3 \left( \cos \frac{\pi}{2} + i \sin \frac{3\pi}{2} \right)
\end{align*}
\]

Convert to rectangular form

polar form \( z = -3(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) \)

rectangular form
\[
\begin{align*}
 a &= r \cos \theta = -3 \cos \frac{3\pi}{4} = -\frac{3\sqrt{2}}{2} \\
 b &= r \sin \theta = -3 \sin \frac{3\pi}{4} = -\frac{3\sqrt{2}}{2} \\
 z &= \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} i
\end{align*}
\]

polar form \( 4(\cos(-120^\circ) + i \sin(-120^\circ)) \)

rectangular form
\[
\begin{align*}
 a &= 4 \cos -120^\circ = 4 \cdot -\frac{1}{2} = -2 \\
 b &= 4 \sin -120^\circ = 4 \cdot -\frac{\sqrt{3}}{2} = -2\sqrt{3} \\
 z &= -2 - 2\sqrt{3} i
\end{align*}
\]
Multiply in rectangular form

Multiply \((2 + 2i\sqrt{3}) \cdot (2 + 2i\sqrt{3})\)

\[-4 - 4i\sqrt{3} + 4i\sqrt{3} + 4i^2\cdot 3\]
\[-4 + -12\]
\[= -16\]

Product should be in same form as factors

Multiply/Divide complex numbers in polar form

\[z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \text{ and } z_2 = r_2(\cos \theta_2 + i \sin \theta_2)\]

1. \[z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))\]

2. \[z_1/z_2 = r_1/r_2 (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))\]
Now multiply again using polar form

Multiply \((-2 + 2\sqrt{3}i) \cdot (2 + 2\sqrt{3}i)\)

\[ z_1 = (-2 + 2\sqrt{3}i) = 4(\cos 120^\circ + i \sin 120^\circ) \]
\[ z_2 = (2 + 2\sqrt{3}i) = 4(\cos 60^\circ + i \sin 60^\circ) \]

\[ z_1 \cdot z_2 = (4 \cdot 4)[\cos (120^\circ + 60^\circ) + i \sin (120^\circ + 60^\circ)] = 16[\cos 180^\circ + i \sin 180^\circ] \]

\(-16 + 0i\)

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Division

\[ 50(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}) \div 5(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) \]

\[ \frac{50}{5} (\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}) \]
\[ -10 - 0i \]

\(-10 + 0i\)
Lesson 6.5
7, 9, 15-23 odd,
29-33 odd, 37-43 odd,
45-51 odd