Lesson 10.4
Mathematical Induction

10.4 Mathematical Induction
A powerful tool for proving all kinds of theorems about positive integers

Let $S_n$ be a statement involving the positive integer $n$.
To prove that $S_n$ is true for all positive integers $n$ requires two steps.

Step 1  Show that $S_1$ is true.
Step 2  Show that if $S_k$ is assumed to be true, then $S_{k+1}$ is also true, for every positive integer $k$. 
The principle of mathematical induction can be illustrated using an unending line of dominoes. If the first domino is pushed over, it knocks down the next, which knocks down the next and so on, in a chain reaction. To topple all the dominoes in the infinite sequence, two conditions must be satisfied:

1. The first domino must be knocked down.
2. If the domino in position \( k \) is knocked down, then the domino in position \( k+1 \) must be knocked down.

http://www.youtube.com/watch?v=WmNczv9jHcg&feature=related
http://www.youtube.com/watch?feature=player_embedded&v=Vp9zLbIE8zo

**Mathematical Induction**

\[
1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}
\]

Step 1: Show that \( S_1 \) is true. 
\[
1 = \frac{1(1+1)}{2} \quad \checkmark
\]

Step 2: Show that if \( S_k \) is true, then \( S_{k+1} \) is true.

\[
1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}
\]

Now add the next term, \( k+1 \), to both sides of \( S_k \)

\[
1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)
\]

\[
1 + 2 + 3 + \ldots + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]

\[
1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2}
\]
Mathematical Induction

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

**Step 1:** Show that \( S_1 \) is true

\[ 1^2 = \frac{1(1+1)(2\cdot1+1)}{6} \quad \checkmark \]

**Step 2:** Show that if \( S_k \) is true, then \( S_{k+1} \) is true.

\[ S_k \Rightarrow 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \]

\[ S_{k+1} \Rightarrow 1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{k+1(k+2)(2k+3)}{6} \]

Now add the next term, \((k+1)^2\), to both sides of \( S_k \)

\[ 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \]

\[ \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \]

\[ = \frac{(k+1)}{6} [k(2k+1) + 6(k+1)] \]

\[ = \frac{(k+1)}{6} (2k^2 + 7k + 6) \]

\[ = \frac{(k+1)(k+2)(2k+3)}{6} \]

This statement is \( S_{k+1} \)
Mathematical Induction \[ 1 + 3 + 6 + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6} \]

**Step 1** Show true for \( n=1 \)

\[ 1 = \frac{1(1+1)(1+2)}{6} \] \( \checkmark \)

**Step 2** \( S_k \rightarrow 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6} \)

\[ S_{k+1} \rightarrow 1 + 3 + 6 + \ldots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \]

\[ \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6} \]

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Mathematical Induction \[ 6 + 10 + 14 + \ldots + (4n+2) = n(2n+4) \]

**Step 1** Show true for \( n=1 \)

\[ 6 = 1(2(1)+4) \] \( \checkmark \)

**Step 2** \( S_k \rightarrow 6 + 10 + 14 + \ldots + 4k+2 = k(2k+4) \)

\[ S_{k+1} \rightarrow 6 + 10 + 14 + \ldots + (4k+2) + (4(k+1)+2) = (k+1)(2(k+1)+4) \]

\[ 6 + 10 + 14 + \ldots + (4k+2) + (4(k+1)+2) = k(2k+4) + (4(k+1)+2) \]

\[ = 2k^2 + 8k + 6 \]

\[ = (2k+6)(k+1) \]
Mathematical Induction
Prove that 2 is a factor of $n^2 + 5n$ for all positive integers $n$

Step 1: Show that $S_1$ is true. $1^2 + 5(1) = 6$, 2 is a factor of 6
Step 2: Show that if $S_k$ is true, then $S_{k+1}$ is true.

$S_k$: 2 is a factor of $k^2 + 5k$
$S_{k+1}$: 2 is a factor of $(k+1)^2 + 5(k+1)$
$S_{k+1}$: $k^2 + 2k + 1 + 5k + 5$
$S_{k+1}$: $k^2 + 7k + 6$
$S_{k+1}$: $(k^2 + 5k) + (2k + 6)$
$S_{k+1}$: divisible by 2    divisible by 2
Mathematical Induction  3 is a factor of $n^3 - 4n$

Mathematical Induction  $1 + 4 + 4^2 + ... + 4^{n-1} = \frac{4^n - 1}{3}$
Mathematical Induction  \[ 4^n - 1 \text{ is divisible by 3} \]

Assignment:

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9, 10, 17-24