11.4 Introduction to Derivatives

Objectives
1. Find slopes and equations of tangent lines.
2. Find the derivative of a function.
3. Find average and instantaneous rates of change.
4. Find instantaneous velocity.

A male’s changing height over intervals of time.

Average growth rates for successively shorter periods of time are shown. Calculus makes these time frames so small that their limit approaches a single point—that is, a single instant in time. This point is shown as point $P$. The slope of the line that touches the graph at $P$ gives the man’s growth rate at one instant in time.

Slopes of Secant Lines

Average Rate of Change

Average Rate of Change
= slope of the line through $(a, f(a))$ and $(b, f(b))$.

\[
\frac{76 - 57}{18 - 13} = 3.8'
\]

The man’s average rate of change, or average growth rate, from 13 to 18 was 3.8 inches per year.
A projectile is propelled into the air from ground level with an initial velocity of 800 ft/sec.

\[ h(t) = -16t^2 + 800t \]

Find the average velocity between:

\[ 10 \leq t \leq 20 \]

\[ \frac{h(20) - h(10)}{20 - 10} = \frac{9600 - 6400}{10} = 320 \]

\[ 20 \leq t \leq 30 \]

\[ \frac{h(30) - h(20)}{30 - 20} = \frac{9600 - 9600}{10} = 0 \]

\[ 30 \leq t \leq 40 \]

\[ \frac{h(40) - h(30)}{40 - 30} = \frac{6400 - 9600}{10} = -320 \]

Average velocity (rate of change) is for some given interval. It is the slope of the secant line through the two given points

<table>
<thead>
<tr>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( h(t_1) )</th>
<th>( h(t_2) )</th>
<th>( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>6400</td>
<td>9600</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>8400</td>
<td>9600</td>
<td>5</td>
</tr>
<tr>
<td>18</td>
<td>20</td>
<td>9216</td>
<td>9600</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>20</td>
<td>9424</td>
<td>9600</td>
<td>1</td>
</tr>
</tbody>
</table>

Notice that as \( \Delta t \) approaches 0, the graph of the line changes from a secant line to a tangent line. The slope of this tangent line is the instantaneous velocity.
Instantaneous Rate of Change

http://www.math.umn.edu/~garrett/qy/Secant.html

http://www.math.umn.edu/~garrett/qy/TraceTangent.html

Tangent Line and Secant Line

Instantaneous rate of change.

If we try to find the rate of change at a point,

\[
\frac{\Delta y}{\Delta x} = 0
\]
Average Rate of Change
Slope of Secant Line

$$\frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$
Instantaneous Rate of Change
Slope of Tangent Line

\[ f(a + h) - f(a) \]
\[ \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

Slope of the Tangent Line to a Curve at a Point

The slope of the tangent line to the graph of a function \( y = f(x) \) at \((a, f(a))\) is given by

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

provided that this limit exists. This limit also describes

- the slope of the graph of \( f \) at \((a, f(a))\).
- the instantaneous rate of change of \( f \) with respect to \( x \) at \( a \).
- Also called the derivative \( f'(x) \)

http://people.hofstra.edu/Stefan_Waner/calctopic1/derivgraph.html
Find the derivative of \( f \) at \( x \). \( f'(x) \)

\[
f(x) = x^2 + 6x + 9
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + 6(x+h) + 9 - (x^2 + 6x + 9)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 6x + 6h + 9 - x^2 - 6x - 9}{h}
\]

\[
= \lim_{h \to 0} \frac{2xh + h^2 + 6h}{h}
\]

\[
= \lim_{h \to 0} \frac{h(2x + h + 6)}{h}
\]

\[
= 2x + 6
\]

\[
f'(x) = 2x + 6
\]

\[
f(x) = x^2 + 6x + 9
\]

Find \( f'(x) \) at \( x = -2 \)

\[
f'(-2) = 2(-2) + 6 = 2
\]

Find \( f'(x) \) at \( x = -5 \)

\[
f'(-5) = 2(-5) + 6 = -4
\]

Find \( f'(x) \) at \( x = -3 \)

\[
f'(-3) = 2(-3) + 6 = 0
\]

Graph It
Finding the Slope of a Tangent Line

Find the slope of the tangent line to the graph of \( f(x) = x^2 + x \) at \((2, 6)\).

\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{2xh + h^2 + h}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{x(2x + h + 1)}{h}
\]

\[
f'(x) = 2x + 1
\]

\[
f'(2) = 2(2) + 1 = 5
\]

The slope of the tangent line to the graph of \( f(x) = x^2 + x \) at \((2, 6)\) is 5.

Find the equation of the tangent line.

\[y = mx + b\]

\[6 = 5(2) + b\]

\[-4 = b\]

\[y = 5x - 4\]
Homework

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15, 19, 21